arXiv:math/0401117v1 [math.CO] 12 Jan 2004

CLASSIFICATION OF FLAG-TRANSITIVE
STEINER QUADRUPLE SYSTEMS

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ABSTRACT. A *Steiner quadruple system* of order v is a $3 - (v, 4, 1)$ design, and will be denoted $SQS(v)$.

Using the classification of finite 2-transitive permutation groups all $SQS(v)$ with a flag-transitive automorphism group are completely classified, thus solving the "still open and longstanding problem of classifying all flag-transitive $3 - (v, k, 1)$ designs" (cf. [5, p. 273], [6]) for the smallest value of k. Moreover, a generalization of a result of H. Lüneburg [14] is achieved.

1. INTRODUCTION

In the last decades, there has been a great interest in classifying $t - (v, k, \lambda)$ designs with certain transitivity properties. For example, all point 2-transitive $2 - (v, k, 1)$ designs were classified by Kantor [12] and a few years later Buekenhout et al. [2] reached a classification of all flag-transitive $2 - (v, k, 1)$ designs. Both results depend on the classification of finite simple groups. However, the classification of flag-transitive $3 - (v, k, 1)$ designs is "a still open and longstanding problem" (cf. [5, p. 273], [6]).

In this article we use the classification of finite 2-transitive permutation groups to classify all flag-transitive $SQS(v)$, thus solving the above problem for the smallest value of k. Moreover, our result generalizes a theorem of Lüneburg [14] that characterizes all flag-transitive $SQS(v)$ under the additional strong assumption that every non-identity element of the automorphism group fixes at most two points. Our procedure as well as our proofs are independent of Lüneburg.

For positive integers $t \le k \le v$ and λ we define a $t - (v, k, \lambda)$ *design* to be an incidence structure $\mathcal{D} = (X, \mathcal{B}, I)$, where X is a set of *points*, $|X| = v$, and $\mathcal{B}$ a set of *blocks*, $|\mathcal{B}| = b$, with the properties that each block $B \in \mathcal{B}$ is incident with k points, and every t-subset of X is incident with λ blocks. A *Steiner quadruple system* of order v, which will be denoted by $SQS(v)$, is a $3 - (v, 4, 1)$ design. Hanani [8] showed that a $SQS(v)$ exists if and only if $v \equiv 2$ or $4 \pmod 6$ $(v \ge 4)$.

In the following let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial $SQS(v)$ and $G \le Aut(\mathcal{D})$ a group of automorphisms of $\mathcal{D}$. A *flag* is an incident point-block pair, that is $x \in X$ and $B \in \mathcal{B}$ such that xIB, and we call $G \le Aut(\mathcal{D})$ to be *flag-transitive* (resp. *block-transitive*) if G acts transitively on the flags (resp. on the blocks) of $\mathcal{D}$. For short, $\mathcal{D}$ is called *flag-transitive (resp. block-transitive, point t-transitive)* if $\mathcal{D}$

admits a flag-transitive (resp. block-transitive, point t-transitive) group of automorphisms.

Our result is the following

Theorem 1. *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a non-trivial $SQS(v)$. Then $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ if and only if one of the following occurs:*

(1) $\mathcal{D}$ *is isomorphic to the $SQS(2^d)$ whose points and blocks are the points and planes of the affine space $AG(d, 2)$, and one of the following holds:*
   (i) $d \geq 3$, *and* $G \cong AGL(d, 2)$,
   (ii) $d = 3$, *and* $G \cong AGL(1, 8)$ *or* $A\Gamma L(1, 8)$,
   (iii) $d = 4$, *and* $G_0 \cong A_7$,
   (iv) $d = 5$, *and* $G \cong A\Gamma L(1, 32)$,

(2) $\mathcal{D}$ *is isomorphic to a $SQS(3^d + 1)$ whose points are the elements of $GF(3^d) \cup \{\infty\}$ and whose blocks are the images of $GF(3) \cup \{\infty\}$ under $PGL(2, 3^d)$ with $d \geq 2$ (resp. $PSL(2, 3^d)$ with $d > 1$ odd) and the derived design is isomorphic to the $2 - (3^d, 3, 1)$ design whose points and blocks are the points and lines of $AG(d, 3)$, and $PSL(2, 3^d) \leq G \leq P\Gamma L(2, 3^d)$,*

(3) $\mathcal{D}$ *is isomorphic to a $SQS(q + 1)$ whose points are the elements of $GF(q) \cup \{\infty\}$ with a prime power $q \equiv 7 \pmod{12}$ and whose blocks are the images of $\{0, 1, \infty, \varepsilon\}$ under $PSL(2, q)$, where ε is a primitive sixth root of unity in $GF(q)$ and the derived design is isomorphic to the Netto triple system,*
   and $PSL(2, q) \leq G \leq P\Sigma L(2, q)$.

A detailed description of the *Netto triple system* can be found in [7, Section 3].

2. Preliminaries

If $\mathcal{D} = (X, \mathcal{B}, I)$ is a $t - (v, k, \lambda)$ design, and $x \in X$ arbitrarily, the *derived* design with respect to x is $\mathcal{D}_x = (X_x, \mathcal{B}_x, I_x)$, where $X_x = X \backslash \{x\}$, $\mathcal{B}_x = \{B \in \mathcal{B} : xIB\}$ and $I_x = I \mid_{X_x \times \mathcal{B}_x}$. We shall also speak of $\mathcal{D}$ as being an *extension* of $\mathcal{D}_x$. Obviously, a derived design is a $(t - 1) - (v - 1, k - 1, \lambda)$ design.

For $g \in G \leq Sym(X)$ let $\text{fix}(g)$ denote the set of fixed points and $\text{supp}(g)$ the support of g. If $\{x_1, ..., x_n\} \subseteq X$ let $G_{\{x_1, ..., x_n\}}$ be its setwise stabilizer and $G_{x_1, ..., x_n}$ its pointwise stabilizer. If $B \in \mathcal{B}$ let G_B be its block stabilizer and $G_{(B)}$ its pointwise block stabilizer. By $r \perp q^n - 1$ we mean that r divides $q^n - 1$ but not $q^k - 1$ for all $1 \leq k < n$.

All other notation is standard.

If $\mathcal{D} = (X, \mathcal{B}, I)$ is a $t - (v, k, 1)$ design then it is elementary that the point 2-transitivity of $G \leq Aut(\mathcal{D})$ implies its flag-transitivity when $t = 2$. However, for $t \geq 3$ the converse holds:

Lemma 2. *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $t - (v, k, 1)$ design with $t \geq 3$. If $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ then G also acts 2-transitively on the points of $\mathcal{D}$.*

Proof. Let $x \in X$. As G acts flag-transitively on $\mathcal{D}$, obviously G_x acts block-transitively on $\mathcal{D}_x$. Since block-transitivity implies transitivity on points for $t \geq 2$ by Block's Theorem [1], G_x also acts transitively on the points of $\mathcal{D}_x$ and the claim follows. $\square$

To classify all flag-transitive $SQS(v)$, we can therefore use the classification of finite 2-transitive permutation groups which itself relies on the classification of finite simple groups (cf. [4], [9], [11], [12]).

The list of groups is as follows:

Let G be a finite 2-transitive permutation group of a non-empty set X. Then we have either

(A) Affine type: G contains a regular normal subgroup T which is elementary abelian of order $v = p^d$, where p is a prime. Let a be a divisor of d. Identify G with a group of affine transformations

$$x \mapsto x^g + c$$

of $V(d, p)$, where $g \in G_0$. Then one the following occurs:

(1) $G \leq A\Gamma L(1, p^d)$
(2) $G_0 \trianglerighteq SL(\frac{d}{a}, p^a)$
(3) $G_0 \trianglerighteq Sp(\frac{2d}{a}, p^a)$, $d \geq 2a$
(4) $G_0 \trianglerighteq G_2(2^a)'$, $d = 6a$
(5) $G_0 \cong A_6$ or A_7, $v = 2^4$
(6) $G_0 \trianglerighteq SL(2, 3)$ or $SL(2, 5)$, $v = p^2$, $p = 5, 7, 11, 19, 23, 29$ or 59, or $v = 3^4$
(7) G_0 contains a normal extraspecial subgroup E of order 2^5, and G_0/E is isomorphic to a subgroup of S_5, where $v = 3^4$
(8) $G_0 \cong SL(2, 13)$, $v = 3^6$,

or

(B) Semisimple type: G contains a simple normal subgroup N, and $N \leq G \leq Aut(N)$. In particular, one of the following holds, where N and $v = |X|$ are given:

(1) A_v, $v \geq 5$
(2) $PSL(d, q)$, $d \geq 2$, $v = \frac{q^d - 1}{q - 1}$, where $(d, q) \neq (2, 2), (2, 3)$
(3) $PSU(3, q^2)$, $v = q^3 + 1$, $q > 2$
(4) $Sz(q)$, $v = q^2 + 1$, $q = 2^{2e+1} > 2$ (Suzuki group)
(5) $^2G_2(q)$, $v = q^3 + 1$, $q = 3^{2e+1} > 3$ (Ree group)
(6) $Sp(2d, 2)$, $d \geq 3$, $v = 2^{2d-1} \pm 2^{d-1}$
(7) $PSL(2, 11)$, $v = 11$
(8) $PSL(2, 8)$, $v = 28$ (N not 2-transitive)
(9) M_v, $v = 11, 12, 22, 23, 24$ (Mathieu group)
(10) M_{11}, $v = 12$
(11) A_7, $v = 15$
(12) HS, $v = 176$ (Higman-Sims group)
(13) Co_3, $v = 276$.   (smallest Conway group)

Let r denote the number of blocks incident with a point. The following obvious observation is important for this paper:

Lemma 3. *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $t - (v, k, 1)$ design, and $x \in X$ arbitrarily. If $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$ then the division property*

$$r \,\Big|\, |G_x|$$

holds.

Counting in two ways easily yields that $r = (v-1)(v-2)/6$ when $\mathcal{D}$ is a $SQS(v)$.

3. Proof of the theorem

Using the notation as before, let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $SQS(v)$. In this section we run through the list of finite 2-transitive permutation groups given in Section 2 and examine successively whether $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$.

3.1. **Affine case.** From Section 2 we know that a 2-transitive permutation group G of affine type has degree $v = p^d$. As a $SQS(v)$ exists if and only if $v = 2$ or $4 \pmod 6$ $(v \geq 4)$ by Hanani's theorem, we conclude that $v = 2^d$ in this case. To avoid trivial $SQS(v)$, let $d \geq 3$.

The following lemma is fundamental for this case.

Lemma 4. *Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $SQS(2^d)$ with $d \geq 3$, and $G \leq Aut(\mathcal{D})$ contains a regular normal subgroup T which is elementary abelian of order $v = 2^d$. If G acts flag-transitively on $\mathcal{D}$ and $|G_0| \equiv 1 \pmod 2$, then $\mathcal{D}$ is uniquely determined (up to isomorphism), and the points and blocks of $\mathcal{D}$ are the points and planes of $AG(d, 2)$.*

Proof. T contains subgroups of order 4 as it is elementary abelian of order 2^d. Moreover, T is the only Sylow 2-group since $|G_0| \equiv 1 \pmod 2$, and contains therefore all subgroups of G of order 4. By assumption, G_B acts transitively on the points of B for $B \in \mathcal{B}$ arbitrarily. Thus 4 is a divisor of the order of G_B, and G_B contains at least one subgroup S of T of order 4. Then $B \in \mathcal{B}$ is an orbit of S and hence an affine plane. As $G \leq Aut(\mathcal{D})$ is block-transitive, we can conclude that all blocks must be affine planes. Now identify the points of $\mathcal{D}$ with the elements of T and the assertion follows. $\square$

Case (1): $G \leq A\Gamma L(1, 2^d)$.

Let $\mathcal{D} = (X, \mathcal{B}, I)$ be a $SQS(2^d)$, $d \geq 3$, and assume $G \leq Aut(\mathcal{D})$ acts flag-transitively on $\mathcal{D}$. Lemma 3 and Lagrange's theorem yield

$$r = \tfrac{1}{3}(2^d - 1)(2^{d-1} - 1) \Big| |G_0| \Big| |A\Gamma L(1, 2^d)_0| = |\Gamma L(1, 2^d)| = d(2^d - 1).$$

Thus $d = 3, 5$. First, assume $d = 3$. Then $|A\Gamma L(1, 8)| = |T| |\Gamma L(1, 8)| = 8 \cdot 7 \cdot 3$. Since G is 2-transitive, we have $8 \cdot 7 \big| |G|$, hence $|G| = 8 \cdot 7$ or $8 \cdot 7 \cdot 3$. The latter implies $G \cong A\Gamma L(1, 8)$, so assume $|G| = 8 \cdot 7$. Since $A\Gamma L(1, 8)$ is solvable, we deduce from Hall's theorem that $G \cong AGL(1, 8)$ as G is a Hall $\{2, 7\}$-group. For $d = 5$ again $|G| = 32 \cdot 31$ or $32 \cdot 31 \cdot 5$. We conclude $G \cong A\Gamma L(1, 32)$ as for $|G| = 32 \cdot 31$ lemma 3 yields a contradiction.

On the contrary, we have to show that $G \cong AGL(1, 8)$, $A\Gamma L(1, 8)$ resp. $A\Gamma L(1, 32)$ acts flag-transitively on the $SQS(8)$ resp. the $SQS(32)$ given in the theorem. For $v = 8$ there exists (up to isomorphism) only the unique $SQS(v)$ consisting of the points and planes of $AG(3, 2)$. Since $G \cong AGL(1, 8)$ acts transitively on the points, it is sufficient to show that $G_0 \cong GL(1, 8)$ acts transitively on the blocks incident with 0. As these are exactly the 2-dimensional subspaces of the underlying vector space, we have

$$B_1 := \{0, 1, t, t + 1\} \neq B_1^t = \{0, t, t^2, t^2 + 1\} \quad \text{for} \quad 1 \neq t \in GL(1, 8) \cong GF(8)^*.$$

Thus $|B_1^{GL(1,8)}| \neq 1$, and hence as $r = 7$, the claim follows by the orbit-stabilizer property. Obviously, $G \cong A\Gamma L(1, 8)$ acts flag-transitively on $\mathcal{D}$ as well. For $v = 32$ we have by lemma 4 also only the unique $SQS(v)$ consisting of the points and planes of $AG(5, 2)$ because $|G_0| = |\Gamma L(1, 32)| \equiv 1 \pmod 2$. To see that

$G_0 \cong \Gamma L(1, 32)$ acts flag-transitively on the blocks incident with 0, examine as before that $|B_1^{GL(1,32)}| \neq 1$, thus $|GL(1,32)_B| = 1$ for any $0 \in B \in \mathcal{B}$ by the orbit-stabilizer property. Hence $|B^{\Gamma L(1,32)}| = 31$ or $31 \cdot 5$. Assuming the first yields $|\Gamma L(1,32)_B| = 5$ by the orbit-stabilizer property again. Let H be a cyclic group of order 5. Then $|H_B| \neq 1$ for any $0 \in B \in \mathcal{B}$. On the other hand, 5 is a 2-primitive divisor of $2^4 - 1$. Thus H has irreducible modules of degree 4 in view of [9, Theorem 3.5]. As the 5-dimensional $GF(32)H$-module is completely reducible by Maschke's theorem, H has as irreducible modules only the trivial module and one of degree 4. But if H fixes any 2-dimensional vector subspace then, again by Maschke's theorem, H would have as irreducible modules two 1-dimensional modules, a contradiction. Therefore, $|B^{\Gamma L(1,32)}| = 31 \cdot 5$ must hold and the claim follows as $r = 31 \cdot 5$.

Case (2): $G_0 \unrhd SL(\frac{d}{a}, 2^a)$.

For $a = 1$ we have $G \cong AGL(d, 2)$. Here G is 3-transitive and the only $SQS(v)$ on which G acts is the one whose points and blocks are the points and planes of $AG(d, 2)$, $d \geq 3$, by Kantor [12]. Obviously, G is also flag-transitive. As $a = d$ has already been done in case (1) we can assume that a is a proper divisor of d. We prove that here no flag-transitive $SQS(v)$ exists.

Because of lemma 3, it is enough to show that r is no divisor of $|G_0|$. Clearly,

$$\left| SL(\tfrac{d}{a}, 2^a) \right| = 2^{d(\frac{d}{a}-1)/2} \prod_{i=2}^{\frac{d}{a}} (2^{ia} - 1),$$

and $\quad [\Gamma L(\tfrac{d}{a}, 2^a) : SL(\tfrac{d}{a}, 2^a)] = |Aut(GF(2^a))| \, |GF(2^a)^*| = a \cdot (2^a - 1).$

Thus it is sufficient to show that r does not divide $a \cdot (2^a - 1) \cdot \left| SL(\tfrac{d}{a}, 2^a) \right|$. By Zsigmondy's theorem (cf. [15, p. 283])

$$2^{d-1} - 1$$

has a 2-primitive prime divisor $\tilde{r} \neq 1$ with $\tilde{r} \perp 2^{d-1} - 1$. Obviously, $\tilde{r} \neq 2$. Furthermore, $\tilde{r} \nmid 3a$ since $\tilde{r} \equiv 1 \ (mod \ (d-1))$ (cf. [9, Theorem 3.5]) and d is properly divisible by a.

Therefore,

$$2^{d-1} - 1 \nmid 3a \cdot 2^{d(\frac{d}{a}-1)/2} \prod_{i=1}^{\frac{d}{a}-1} (2^{ia} - 1)$$

and the claim follows.

Cases (3)-(4): These cases can be eliminated analogous case (2) using lemma 3 and Zsigmondy's theorem. (For $|Out(G_0)|$ see e.g. [13, Table 5.1 A]).

Case (5): $G_0 \cong A_6$ or A_7, $v = 2^4$.

If $G \cong A_6$ then lemma 3 implies that G cannot act flag-transitively on any $SQS(v)$.

As $G \cong A_7$ is 3-transitive and the only $SQS(v)$ on which G acts is the one whose points and blocks are the points and planes of $AG(4, 2)$ by Kantor [12], we have also flag-transitivity in this case.

Cases (6)-(8): These cases cannot occur since v is no power of 2.

3.2. **Semisimple case.** The cases $(3), (5), (8), (12)$ from the list where G is of semisimple type can easily be ruled out as above by using lemma 3. Obviously, the cases $(4), (7), (10), (11), (13)$ cannot occur by Hanani's theorem.

Before we proceed we indicate

Lemma 5. *Let $V(d, q)$ be a vector space of dimension $d > 3$ over $GF(q)$ and $PG(d-1, q)$ the $(d-1)$-dimensional projective space. Assume G containing $PSL(d, q)$ acts on $PG(d-1, q)$ and for all $g \in G$ with $|M^g \cap M| \geq 3$ we have $M^g = M$, where M is an arbitrary set of points of $PG(d-1, q)$ of cardinality k with $3 \leq k \leq |H|$, and H a hyperplane of $PG(d-1, q)$.*

If $|M \cap H| \geq 3$, then $M \cap H = M$ holds.

Proof. For $k = 3$ the assertion is trivial. So assume $3 < k \leq |H| = \frac{q^{d-1}-1}{q-1}$. In $PG(d-1, q)$ Desargues' theorem holds and the translations $T(H)$ form an abelian group which is sharply transitive on the points of $PG(d-1, q) \setminus H$ by Baer's theorem. But on H the group $T(H)$ acts trivially since the central collineations fix each point of H. Thus the claim holds if all elements of M lie in H. Therefore, assume that there is an element of M which is not in H. Then M contains all points of $PG(d-1, q) \setminus H$ as $T(H)$ is transitive. Thus

$$|M| \geq \frac{q^d - 1}{q - 1} - \frac{q^{d-1} - 1}{q - 1} = \frac{q^d - q^{d-1}}{q - 1} = q^{d-1} > \frac{q^{d-1} - 1}{q - 1} = |H|.$$

But this contradicts the assumption $|M| \leq |H|$, and the claim follows. $\square$

Case (1): $N = A_v$, $v \geq 5$. Here, G is 3-transitive and does not act on any non-trivial $3 - (v, k, 1)$ design by Kantor [12].

Case (2): $N = PSL(d, q)$, $d \geq 2$, $v = \frac{q^d - 1}{q - 1}$, where $(d, q) \neq (2, 2), (2, 3)$.

We distinguish two subcases:

(i) $N = PSL(2, q)$, $v = q + 1$.

Here $q \geq 5$ as $PSL(2, 4) \cong PSL(2, 5)$, and $Aut(N) = P\Gamma L(2, q)$. First suppose that G is 3-transitive. According to Kantor [12], we have then only the $SQS(3^d + 1)$ described in (2) of theorem 1, and $PSL(2, 3^d) \leq G \leq P\Gamma L(2, 3^d)$. Obviously, also flag-transitivity holds. As $PGL(2, q)$ is a transitive extension of $AGL(1, q)$, it is easily seen that the derived design at any point of $GF(3^d) \cup \{\infty\}$ is isomorphic to the $2 - (3^d, 3, 1)$ design consisting of the points and lines of $AG(d, 3)$.

Now assume that G is 3-homogeneous but not 3-transitive. As here $PSL(2, q)$ is a transitive extension of $AG^2 L(1, q)$ we deduce from [7] that the derived design is either the affine space $AG(d, 3)$ or the Netto triple system. Thus (2) with the part in brackets or (3) of theorem 1 holds with $PSL(2, 3^d) \leq G \leq P\Sigma L(2, 3^d)$ (where $P\Sigma L(2, p^d) := PSL(2, p^d) \rtimes <\tau_\alpha>$ with $\tau_\alpha \in Sym(GF(p^d) \cup \{\infty\}) \cong S_v$ of order d induced by the Frobenius automorphism $\alpha : GF(p^d) \longrightarrow GF(p^d)$, $x \mapsto x^p$.) Conversely, as G is 3-homogeneous it is also block-transitive. In both cases we have $PSL(2, q)_B \cong A_4$ for any $B \in \mathcal{B}$ since $PSL(2, q)_B$ has order 12 by the orbit-stabilizer property and $PSL(2, q)_B \longrightarrow Sym(B) \cong S_4$ is a faithful representation. Thus, in each case flag-transitivity holds.

Finally, suppose G is not 3-homogeneous. As $PGL(2, q)$ is 3-homogeneous the $PGL(2, q)$-orbit on 3-subsets therefore splits under $PSL(2, q)$ into two orbits of same length. Let M be an arbitrary 3-subset. Then $|PSL(2, q)_M| = |PGL(2, q)_M| = 6$ by the orbit-stabilizer property. Thus, as $PGL(2, q)$ is 3-transitive

we have $PSL(2,q)_M \cong S_3$ for each orbit. If $PSL(2,q)$ acts block-transitively on any $SQS(v)$ then $PSL(2,q)_B \cong A_4$ again for any $B \in \mathcal{B}$. But, by the definition of $SQS(v)$ this would imply that $PSL(2,q)_{\tilde{B}}$, where $\tilde{B}$ denotes the block uniquely determined by M, contains $PSL(2,q)_M$, a contradiction. Thus $PSL(2,q)$ does not act flag-transitively on any $SQS(v)$. We show now that G cannot act flag-transitively on any $SQS(v)$. Without restriction choose $\mathcal{O}_1$ to be the $PSL(2,q)$-orbit containing $\{0,1,\infty\}$. Easy calculation shows that $P\Sigma L(2,q)_{0,1,\infty} = <\tau_\alpha>$. Thus $P\Sigma L(2,q)_{\mathcal{O}_1}$ is contained in $P\Gamma L(2,q)$, and equality holds as $P\Sigma L(2,q)$ is of index 2 in $P\Gamma L(2,q)$ and $P\Gamma L(2,q)$ is 3-transitive. Therefore, we only have to consider $PSL(2,q) \leq G \leq P\Sigma L(2,q)$. Dedekind's law yields $G = PSL(2,q) \rtimes (G \cap <\tau_\alpha>)$ and $G_{(B)} = PSL(2,q)_{(B)} \rtimes G \cap <\tau_\alpha> = G \cap <\tau_\alpha> \cong C_m$, the cyclic group of order $m \mid d$, for any $B \in \mathcal{B}$ since every non-identity element of $PSL(2,q)$ fixes at most two points. Assume G acts block-transitively on any $SQS(v)$. Then we can choose $B \in \mathcal{B}$ such that B contains $\{0,1,\infty\}$. Since $G_{(B)}$ is the kernel of the representation $G_B \longrightarrow Sym(B) \cong S_4$ and $PSL(2,q)_B \cong A_4$ we have therefore again by Dedekind's law

$$G_B = PSL(2,q)_B \times (G \cap <\tau_\alpha>) \cong A_4 \times C_m.$$

However, as $PSL(2,q)_{\{0,1,\infty\}} \cong S_3$ we get analogously

$$G_{\{0,1,\infty\}} = PSL(2,q)_{\{0,1,\infty\}} \times (G \cap <\tau_\alpha>) \cong S_3 \times C_m,$$

which leads again to a contradiction by the definition of $SQS(v)$.

(ii) $N = PSL(d,q)$, $d \geq 3$, $v = \frac{q^d-1}{q-1}$.

Here $Aut(N) = P\Gamma L(d,q) \rtimes <\iota>$, where ι denotes a graph automorphism. We show that G does not act on any $SQS(v)$. For $d = 3$ this is obvious since $v = q^2 + q + 1$ is always odd, a contradiction to Hanani's theorem.

Consider $d > 3$ and let H be a hyperplane of the projective space $PG(d-1,q)$. Assume that the claim does not hold. Then there is a counterexample with d minimal. Without restriction we can choose three arbitrary points α, β, γ from H. As for $d > 3$

$$|H| = \frac{q^{d-1}-1}{q-1} > 4$$

holds, the block uniquely determined by α, β, γ is contained in H by lemma 5. Thus H induces a $SQS(\frac{q^{d-1}-1}{q-1})$ on which G containing $PSL(d-1,q)$ operates. By induction, we get the minimal counterexample for $d = 3$. So G containing $PSL(3,q)$ acts on a $SQS(\frac{q^3-1}{q-1})$. But, as above $\frac{q^3-1}{q-1} = q^2 + q + 1$ is always odd yielding the desired contradiction.

Case (6): $N = Sp(2d,2)$, $d \geq 3$, $v = 2^{2d-1} \pm 2^{d-1}$.

Here $N = G$ since $|Out(N)| = 1$ (cf. [13, Table 5.1 A]). We show that G contains elements which fix exactly 3 points and hence cannot act on any $SQS(v)$ by definition.

Let X^+ respectively X^- denote the set of points on which G operates with $|X^+| = 2^{2d-1} + 2^{d-1}$ resp. $|X^-| = 2^{2d-1} - 2^{d-1}$, and define

$$m_p(G) := \min\{|\text{supp}(g)| : 1 \neq g \in G, g \text{ a } p\text{-element of } G\}$$

to be the *minimal p-degree* of a transitive permutation group G, p a prime divisor of $|G|$ (cf. [10]).

First, suppose d is even. By Zsigmondy's theorem

$$2^{d-1} - 1$$

has a 2-primitive prime divisor p with $p \perp 2^{d-1} - 1$. Moreover, p divides $|G|$ since $|G| = 2^{d^2} \prod_{i=1}^{d}(2^{2i} - 1)$ (see e.g. [13, Table 2.1 C]). Therefore, according to [10, Theorem 3.7] we get in X^+

$$m_p(G) = 2^{2d-2(d-1)-1}(2^{2(d-1)} - 1) + 2^{d-(d-1)-1}(2^{d-1} - 1) = \left|X^+\right| - 3.$$

Thus, there exists $g \in G$ of prime order p that fixes 3 points in X^+.

For $d \neq 4$ Zsigmondy's theorem yields the existence of a 2-primitive prime divisor p with $p \perp 2^{2(d-1)} - 1$ and as p divides $|G|$ we have in X^- again by [10, Theorem 3.7]

$$m_p(G) = 2^{2d-2(d-1)-1}(2^{2(d-1)} - 1) - 2^{d-(d-1)-1}(2^{d-1} + 1) = \left|X^-\right| - 3.$$

When $d = 4$ then [3, p. 123] yields $|\text{fix}(g)| = 3$ in X^- for $g \in 3D$, where $3D$ denotes a conjugacy class in [3].

Now, suppose d is odd. Again by Zsigmondy's theorem and [10, Theorem 3.7] there exists a 2-primitive prime divisor p with $p \perp 2^{2(d-1)} - 1$, and $m_p(G) = |X^-| - 3$ in X^-.

If $d \neq 7$ Zsigmondy's theorem yields the existence of a 2-primitive prime divisor p with $p \perp 2^{d-1} - 1$. Choose $\begin{pmatrix} A_0 & A_1 \\ A_2 & A_3 \end{pmatrix} \in S \in Syl_p(Sp(d - 1, 2))$ and define

$$h := \begin{pmatrix} A_0 & & & A_1 & & \\ & A_0 & & & A_1 & \\ & & 1 & & & 0 \\ A_2 & & & A_3 & & \\ & A_2 & & & A_3 & \\ & & 0 & & & 1 \end{pmatrix}.$$

The proof of [10, Theorem 3.7] yields $|\text{fix}(h)| = 3$ in X^+ and $|\text{fix}(h)| = 1$ in X^-. For $d = 7$ choose $A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and define $k := diag(A, A, A, 1, {}^tA^{-1}, {}^tA^{-1}, {}^tA^{-1}, 1)$. Again, $|\text{fix}(k)| = 3$ in X^+ and $|\text{fix}(k)| = 1$ in X^-. Thus the assertion is proved.

  Case (9): M_v, $v = 11, 12, 22, 23, 24$.

  Here, only $v = 22$ is possible by Hanani's theorem. But as M_{22} is 3-transitive, Kantor [12] shows that the only $3 - (v, k, 1)$ design on which M_{22} resp. $Aut(M_{22})$ acts is the $3 - (22, 6, 1)$ design. Therefore, this case cannot occur finishing the proof of theorem 1.

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